
UNIT 4 MATHEMATICAL INDUCTION

Structure

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4.0 INTRODUCTION

We begin with the following question. What is the sum of first n odd natural numbers ?

If n equals 1, the sum equals 1, as 1 is the only summand. The answer we seek is a formula that will enable us to determine this sum for each value n without having to add the summands.

Table 4.1 lists the sum S_n of the first n odd natural numbers, as n takes values from 1 to 10.

Table 4.1

n	<i>Series</i>	<i>Sum (S_n)</i>
1	1	$1=1^2$
2	1+3	$4=2^2$
3	1+3+5	$9=3^2$
4	1+3+5+7	$16=4^2$
5	1+3+5+7+9	$25=5^2$
6	1+3+5+7+9+11	$36=6^2$
7	1+3+5+7+9+11+13	$49=7^2$
8	1 + 3 ++15	$64=8^2$
9	1 + 3 ++17	$81=9^2$
10	1 + 3 ++19	$100=10^2$

Jumping to a Conclusion

Judging from the pattern formed by first 10 sums, we might conjecture that

$$S_n = 1 + 3 + 5 + \dots + (2n - 1) = n^2.$$

Recognizing a pattern and then simply jumping to the conclusion that the pattern must be true for all values of n is **not a logically valid method** of proof in

mathematics. There are many instances when the pattern appears to be developing for small values of n and then at some point the pattern fails. Let us look at one example. It was widely believed that $P_n = n^2 + n + 41$ is prime for all natural-numbers. Indeed p_n is prime for all values of n lying between 1 and 39 as shown in Table 4.2.

But the moment we take $n = 40$, we get

$$\begin{aligned} P_{40} &= 40^2 + 40 + 41 \\ &= 1600 + 40 + 41 = 1681 = 41^2 \end{aligned}$$

which is clearly not a prime.

Table 4.2

n	P_n	n	P_n	n	P_n
1	43	11	173	26	743
		12	197	27	797
2	47	13	223	28	853
3	53	14	251	29	911
		15	281	30	971
4	61	16	313	31	1033
5	71	17	347	32	1097
		18	383	33	1163
6	83	19	421	34	1231
7	97	20	461	35	1301
				36	1373
8	113	21	503	37	1447
9	131	22	547	38	1523
		23	593		
10	151	24	641	39	1601
		25	691		

Just because a rule, pattern or formula seems to work for several values of n , we cannot simply conclude that it is valid for all values of n without going through a *legitimate proof*.

How to Legalize a Pattern?

One way to legalize the pattern is to use the principle of **Mathematical induction**. To see what it is, let us return to our question in the beginning of the chapter. What is the sum of first n odd natural numbers?

We have already seen that the formula

$$S_n = 1 + 3 + 5 + \dots + (2n - 1) = n^2 \text{ is valid for } n = 1, 2, 3, \dots, 10 \quad (1)$$

Do we need to compute S_n by adding the first n odd natural numbers ?

A moment's reflection will show that it is not necessary.

Having obtained the value of S_n for some integer n , we can obtain the value of

$$S_{n+1} = S_n + 2n + 1$$

if $S_n = n^2$ for some n , then $S_{n+1} = S_n + 2n + 1 = n^2 + 2n + 1 = (n+1)^2$.

That is, if $S_n = n^2$ for some natural number n , then the formula holds for the next natural number $n + 1$.

Since the formula $S_n = n^2$ holds for $n = 10$, therefore it must hold $n = 11$. Since, it holds for $n = 11$, therefore, it must hold for $n = 12$. Since, it holds for $n = 12$, it holds for $n = 13$, and so on. The principle underlying the foregoing argument is nothing but the principle of mathematical induction. We state this formally in section 4.3.

4.1 OBJECTIVES

After studying this unit, you should be able to:

- use the principle of mathematical induction to establish truth of several formulae and inequalities for each natural number n .

4.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

Let P_n be a statement involving the natural number n . If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1} , for every integer k , then P_n must be true all natural numbers n .

In other words, to prove that a statement P_n holds for all natural numbers, we must go through *two* steps; First, we must prove that P_1 is true. Second, we must prove that P_{k+1} is true whenever P_k is true.



Just proving P_{k+1} whenever P_k is true will not work.

There is an interesting analogue. Suppose we have “sequence” of dominoes standing in a row, as in Fig. 4.1 Suppose (1) the first domino falls, and (2) whenever any domino falls, then the one next to it (to the right in Fig. 4.2) falls as well. Our conclusion is that each domino will fall (see Fig 4.3). This reasoning closely parallels the ideal of induction.

To apply the principles of mathematical induction, we always need to be able to find P_{k+1} for a given P_k . It is important to acquire some skill in writing P_{k+1} whenever P_k is given.

We now take up some illustrations in which we write some particular terms when we know P_n . We also take up some illustrations in which we write P_{k+1} when we know P_k .

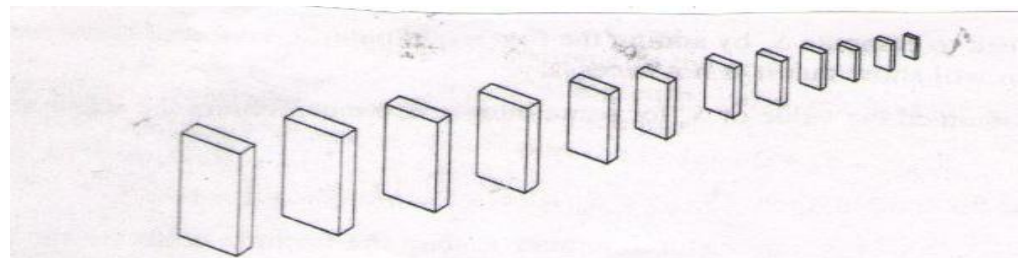


Figure 4.1

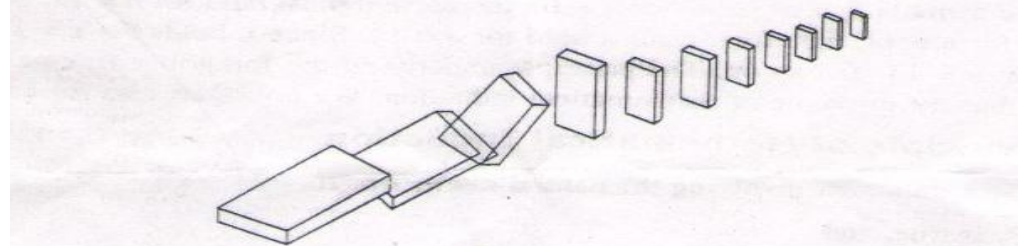


Figure 4.2

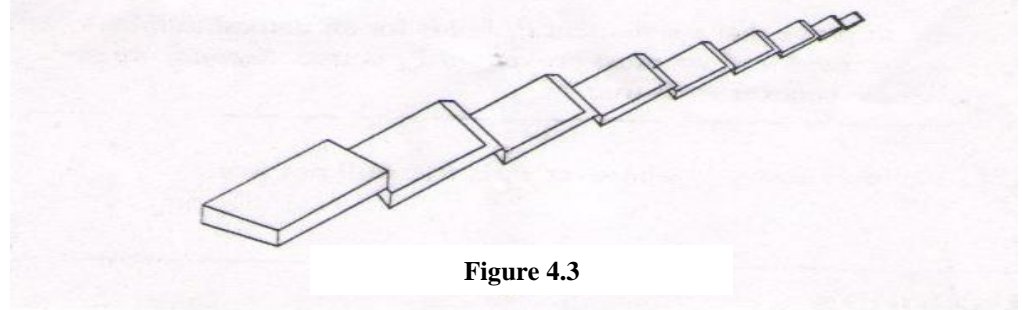


Figure 4.3

Illustration 1 : If P_n is the statement “ $n(n + 1)$ is even”, then what is P_4 ? What is P_{10} ?

Solution :

P_4 is the statement “ $4(4+1)$ is even”, i.e., “20 is even”. P_{10} is the statement “ $10(10+1)$ is even” i.e., “110 is even”.

Illustration 2 : If P_n is the statement “ $n(n+1)(n+2)$ is divisible by 12”, write P_3 , P_4 and P_5 . Which one of P_3 , P_4 and P_5 are true statement ?

Solution

P_3 is “3 (3+1) (3+2) is divisible by 12” i.e., “60 is divisible by 12”

P_4 is “4 (4+1) (4+2) is divisible by 12” i.e., “120 is divisible by 12”

P_5 is “5 (5+1) (5+2) is divisible by 12” i.e., “210 is divisible by 12”

Each of P_3 and P_4 is true. But P_5 is false.

Example 1

- (i) If P_n is the statement “ $n^3 + n$ is divisible by 3”, is the statement P_3 true ? Is the statement P_4 true ?
- (ii) If P_n is the statement “ $2^{3n} - 1$ is an integral multiple of 7”, prove that P_1 , P_2 and P_3 are true ?
- (iii) If P_1 is the statement “ $3^n > n$ ” are true P_1 , P_2 and P_4 true statements ?
- (iv) If P_n is the statement “ $2^n > n$ ” what is P_{n+1} ?
- (v) If P_n is the statement “ $3^n > n$ ” prove that P_{n+1} is true whenever P_n is true.
- (vi) Let P_n is the statement “ $n^2 > 100$ ” prove that P_{k+1} is true whenever P_k is true.
- (vii) If P_n is the statement “ $2^n > 3n$ ” and if P_k is true, prove that P_{k+1} is true.
- (viii) If P_n is the statement “ $2^{3n} - 1$ is a multiple of 7”, prove that truth of P_k implies the true of P_{k+1} .
- (ix) If P_n is the statement “ $10^{n+1} > (n+1)^5$ ”, prove that P_{k+1} is true whenever P_k is true.
- (x) Give an example of a statement P_n , such that P_3 is true but P_4 is not true.
- (xi) Give an example of statement P_n such that it is not true for any n .
- (xii) Give an example of a statement P_n in which P_1 , P_2 , P_3 are not true but P_4 is true.
- (xiii) Give an example of a statement P_n which is true for each n .

Solution :

- (i) P_3 is the statement “ $3^3 + 3$ is divisible by 3” i.e., “30 is divisible by 3”. which is clearly true.
 P_4 is the statement “ $4^3 + 4$ is divisible by 4” i.e., “68 is divisible by 3”
This is clearly not true.
- (ii) P_1 is the statement “ $2^3 - 1$ is an integral multiple of 7”, i.e., “7 is an integral multiple of 7”. This is a true statement.

P_2 is the statement “ $2^6 - 1$ is an integral multiple of 7”, i.e., 63 is an integral multiple of 7”. This also is a true statement.

P_3 is the statement “ $2^9 - 1$ is an integral multiple of 7”, i.e., “511 is an integral multiple of 7”. This again is a true statement.

(iii) P_1 is $3^1 > 1$, which is clearly true.

P_2 is “ $3^2 > 2$ ”. This also is a true statement.

P_4 is “ $3^4 > 4$ ”. This again is a true statement.

(iv) P_{n+1} is the statement “ $2^{n+1} > n+1$ ”.

(v) We are given that $3^n > n$.

we are interested to show that $3^{n+1} > n+1$

$$\text{we have } \frac{n+1}{n} = 1 + \frac{1}{n} \leq 1 + 1 < 3$$

$$\Rightarrow n+1 < 3n < 3 \cdot 3^n = 3^{n+1}.$$

This show that if P_n is true, then P_{n+1} is true.

(vi) We are given that $k^2 > 100$.

we wish to show that $(k+1)^2 > 100$

we have

$$(k+1)^2 = k^2 + 2k + 1 > k^2 > 100 \quad [\because 2k+1 > 0]$$

$$\Rightarrow (k+1)^2 > 100.$$

This shows that P_{k+1} is true whenever P_k is true.

(vii) Since P_k is true, we get $2^k > 3k$.

we wish to show that $2^{k+1} > 3(k+1)$

we have

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k = 2^k + 2^k > 3k + 3k && [\text{by assumption}] \\ &> 3k + 3 \end{aligned}$$

$$\Rightarrow 2^{k+1} > 3(k+1)$$

this proves that P_{k+1} is true.

(viii) Since P_k is true we have $2^{3k} - 1$ is a multiple of 7, i.e., there exists an integer m such that $2^{3k} - 1 = 7m$

We wish to show that $2^{3(k+1)} - 1$ is a multiple of 7.

We have

$$\begin{aligned} 2^{3(k+1)} - 1 &= 2^{3k} \cdot 2^3 - 1 = (7m+1) \cdot (8) - 1 \\ &= 56m + 8 - 1 = 56m + 7 = 7(8m + 1) \end{aligned}$$

This shows that $2^{3(k+1)} - 1$ is a multiple of 7, i.e. P_{k+1} is true.

(ix) Since P_k is true, we have $10^{k+1} > (k+1)^5$

We wish to show that $10^{k+2} > (k+2)^5$

We have

$$\frac{(k+2)^5}{(k+1)^5} = \left(1 + \frac{1}{k+1}\right)^5$$

$$\text{As } k \geq 1, k+1 \geq 2 \Rightarrow \frac{1}{k+1} \leq \frac{1}{2}$$

$$\text{therefore, } \left(1 + \frac{1}{k+1}\right)^5 \leq \left(1 + \frac{1}{2}\right)^5 = \left(\frac{3}{2}\right)^5 = \frac{243}{32} < 10$$

$$\text{Thus, } (k+2)^5 < 10 (k+1)^5 < 10 \cdot 10^{k+1} = 10^{k+2}.$$

Therefore, P_{k+1} is true.

(x) Let P_n be that statement “ $n \leq 3$ ”, then P_3 is true but P_4 is not true.

(xi) Let P_n be the statement “ $n(n+1)$ is odd”. Then P_n is false for every n .

(xii) Let $P_n, n \geq 4$.

(xiii) Let P_n be the statement “ $n \geq 1$ ”. The P_n is true for each n .

Example 2: Use the principle of mathematical induction to prove that

$$2 + 4 + 6 + \dots + 2n = n(n+1)$$

for each natural number n .

Solution :

Mathematical induction consists of two distinct parts. First, we must show that the formula holds for $n = 1$.

Let P_n denote the statement

$$2 + 4 + 6 + \dots + 2n = n(n+1)$$

Step 1. When $n=1$, P_n becomes $2 = 1(1+1)$

which is clearly true.

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for *some* integer k . The second step is to use this assumption to prove that formula is valid for the next natural number $k+1$.

Step 2. Assume that P_k is true for some $k \in \mathbf{N}$, that is, assume that

$$2 + 4 + 6 + \dots + 2k = k(k+1)$$

is true. We must show that P_{k+1} is true, where P_{k+1}

$$2 + 4 + 6 + \dots + 2k + 2(k+1) = (k+1)(k+2) \quad (1)$$

Not to Forget

While writing LHS of P_{k+1} , you must remember that not only should you write the last term of the series, but also a term prior to the last term. If you, now suppress the last term of the LHS of P_{k+1} what remain of the LHS of P_k .

$$\begin{aligned}
 \text{LHS of (1)} &= 2 + 4 + 6 + \dots + 2k + 2(k+1) \\
 &= k(k+1) + 2(k+1) && [\text{induction assumption}] \\
 &= (k+1)(k+2) && [\text{taking } k+1 \text{ common}] \\
 &= \text{RHS of (1)}
 \end{aligned}$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

CAUTION

You will lose at least one mark if you do not write this last paragraph.

Example 3 Use the principle of mathematical induction to show that

$$1 + 4 + 7 + \dots + (3n-2) = \frac{1}{2}n(3n-1)$$

for every natural number n .

Solution : Let P_n denote the statement.

$$1 + 4 + 7 + \dots + (3n-2) = \frac{1}{2}n(3n-1)$$

When $n = 1$, P_n becomes $1 = \frac{1}{2}(1)[3(1) - 1]$ or $1 = 1$

which is clearly true.

This shows that the result holds $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$. That is, assume that

$$1 + 4 + 7 + \dots + (3k-2) = \frac{1}{2}k(3k-1)$$

We shall now show that the truth of P_k implies the true of P_{k+1} where P_{k+1} is

$$1 + 4 + 7 + \dots + (3k-2) + \{3(k+1) - 2\} = \frac{1}{2}(k+1)(3(k+1) - 1)$$

$$\text{or } 1 + 4 + 7 + \dots + (3k-2) + (3k+1) = \frac{1}{2}(k+1)(3k+2) \quad (1)$$

LHS of (1)

$$= 1 + 4 + 7 + \dots + (3k-2) + (3k+1)$$

$$= \frac{1}{2}k(3k-1) + (3k+1) \quad [\text{induction assumption}]$$

$$\begin{aligned}
 &= \frac{1}{2} [3k^2 - k + 6k + 2] \\
 &= \frac{1}{2} (3k^2 + 5k + 2) \\
 &= \frac{1}{2} [3k^2 + 3k + 2k + 2] \\
 &= \frac{1}{2} [3k(k+1) + 2(k+1)] \\
 &= \frac{1}{2} (k+1)(3k+2) = \text{RHS of (1)}
 \end{aligned}$$

This shows that the result holds for $n = k + 1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Example 4 : Use the principle of mathematical induction to prove that

$$1^3 + 2^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2 \text{ for every natural number } n.$$

Solution: Let P_n denote the statement

$$1^3 + 2^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

When $n = 1$, P_n becomes

$$1^3 + 2^3 + \dots + n^3 = \frac{1}{4}(1^2)(1+1)^2 \text{ or } 1 = 1$$

This shows that the result holds for $n = 1$. Assume that P_k is true for some $k \in \mathbf{N}$.

That, is assume that

$$1^3 + 2^3 + \dots + k^3 = \frac{1}{4}k^2(k+1)^2$$

We shall now show that the truth of P_k implies the truth of P_{k+1} where P_{k+1} is

$$1^3 + 2^3 + \dots + k^3 = \frac{1}{4}(k+1)^2(k+1)^2 \quad (1)$$

LHS of (1)

$$= 1^3 + 2^3 + \dots + k^3 = \frac{1}{4}k^2(k+1)^2$$

$$= \frac{1}{4}k^2(k+1)^2(k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2(k+1)^3 \quad [\text{induction assumption}]$$

$$= \frac{1}{4}(k+1)^2[k^2 + 4k(k+1)] = \frac{1}{4}(k+1)^2(k+2)^2$$

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Example 5 : Use the principle of mathematical induction to prove that

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for every natural number n .

Solution: Let P_n denote the statement

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

When $n = 1$, P_n becomes $\frac{1}{1.2} = \frac{1}{1+1}$ or $\frac{1}{2} = \frac{1}{2}$

This shows that the result holds for $n = 1$. Assume that P_k is true for some $k \in \mathbf{N}$.

That, is assume that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad (1)$$

We shall now show that the true of P_k implies the truth of P_{k+1} where P_{k+1} is

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} = \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\text{LHS of (1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad [\text{induction assumption}]$$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

= RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Example 6 : Use the principle of mathematical induction to show that

$$2 + 2^2 + \dots + 2^n = 2^{n+1} - 2$$

for every natural number n .

Solution : Let P_n denote the statement

$$2 + 2^2 + \dots + 2^n = 2^{n+1} - 2$$

When $n = 1$, P_n becomes

$$2 = 2^{1+1} - 2 \text{ or } 2 = 4 - 2$$

This shows that the result holds for $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$.

That is, assume that

$$2 + 2^2 + \dots + 2^k = 2^{k+1} - 2$$

We shall now show that truth of P_k implies the truth of P_{k+1} is

$$2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+1} - 2 \quad (1)$$

$$\text{LHS of (1)} = 2 + 2^2 + \dots + 2^k + 2^{k+1}$$

$$= (2^{k+1} - 2) + 2^{k+1} \quad [\text{induction assumption}]$$

$$= 2^{k+1} (1 + 1) - 2$$

$$= 2^{k+1} 2 - 2 = 2^{k+2} - 2$$

$$= \text{RHS of (1)}$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Example 7: Show that $2^{3n} - 1$ is divisible by 7 for every natural number n .

Solution : Let P_n denote the statement $7|(2^{3n} - 1)$

For $n = 1$, P_n becomes $7|(2^3 - 1)$

Since $2^3 - 1 = 8 - 1 = 7$, we have $7|7$. This shows that the result is true for $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$.

That is, assume that $7|(2^{3k} - 1)$

That is, assume that $2^{3k} - 1 = 7m$ for some $m \in \mathbf{N}$.

We shall now show that the truth of P_k implies the truth of P_{k+1} , where P_{k+1} is

$$7|(2^{3(k+1)} - 1)$$

Now

$$2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 2^{3k} \cdot 2^3 - 1$$

$$= (7m + 1)(8) - 1 \quad [\because 2^{3k} - 1 = 7m]$$

$$= 56m + 8 - 1 = 56m + 7 = 7(8m + 1)$$

$$\Rightarrow 7|[2^{3(k+1)} - 1]$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Example 8 Show that $n(n+1)(2n+1)$ is a multiple of 6 for every natural number n .

Solution : Let P_n denote the statement $n(n+1)(2n+1)$ is a multiple of 6.

When $n=1$, P_n becomes $1(1+1)((2)(1)+1) = (1)(2)(3) = 6$ is a multiple of 6.

This shows that the result is true for $n = 1$.

Assume that P_k is true for some $k \in \mathbf{N}$. That is assume that $k(k+1)(2k+1)$ is a multiple of 6.

Let $k(k+1)(2k+1) = 6m$ for some $m \in \mathbf{N}$.

We now show that the truth of P_k implies the truth of P_{k+1} , where P_{k+1} is

$(k+1)(k+2)[2(k+1)+1] = (k+1)(k+2)(2k+3)$ is a multiple of 6.

We have

$$\begin{aligned} & (k+1)(k+2)(2k+3) \\ &= (k+1)(k+2)[(2k+1)+2] \\ &= (k+1)[k(2k+1)+2(2k+1)+4] \\ &= (k+1)[k(2k+1)+6(k+1)] \\ &= k(k+1)(2k+1)+6(k+1)^2 \\ &= 6m+6(k+1)^2 = 6[m+(k+1)^2] \end{aligned}$$

Thus $(k+1)(k+2)(2k+3)$ is multiple of 6.

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Example 9 : Show that 11 divides $10^{2n-1} + 1$ for every natural number n .

Solution : Let P_n denote the statement

$$11|(10^{2n-1} + 1).$$

When $n = 1$, P_n becomes $11|(10^{2-1} + 1)$.

As $10^{2-1} + 1 = 10 + 1 = 11$ we have $11|11$.

This shows that the result is true for $n = 1$

Assume that P_k is true for some $k \in \mathbf{N}$. That is, assume that

$$11(10^{2k-1} + 1).$$

That is, assume that $10^{2k-1} + 1 = 11m$ for some $m \in \mathbf{N}$.

We shall now that the truth of P_k implies P_{k+1} , where P_{k+1} is

$$11(10^{2(k+1)-1} + 1)$$

$$\begin{aligned} \text{Now, } 10^{2(k+1)-1} + 1 &= 10^{2k+2-1} + 1 \\ &= 10^{2k-1+2} + 1 = 10^{2k-1} \cdot 10^2 \\ &= (11m - 1) 10^2 + 1 \quad [\because 10^{2k-1} + 1 = 11m] \\ &= 1100m - 100 + 1 = 1100m - 99 = 11(100m - 9) \\ \Rightarrow &11 \mid (10^{2(k+1)-1} + 1). \end{aligned}$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Example 10 Show that 133 divides $11^{n+2} + 12^{2n+1}$ for every natural number n .

Solution Let P_n denote the statement

$$133 \mid (11^{n+2} + 12^{2n+1})$$

When $n = 1$, P_n becomes $133 \mid (11^{1+2} + 12^{2+1})$.

$$\begin{aligned} \text{As } 11^{1+2} + 12^{2+1} &= 11^3 + 12^3 = 1331 + 1728 = 3059 \\ &= (133)(29), \text{ we have } 133 \mid (11^{1+2} + 12^{2+1}) \end{aligned}$$

This shows that the result is true for $n = 1$. Assume that P_k is true for some $k \in \mathbf{N}$. That is assume that

$$133 \mid (11^{k+2} + 12^{2k+1})$$

That is, assume that $11^{k+2} + 12^{2k+1} = 133m$ for some $m \in \mathbf{N}$. We shall now show that the truth of P_k implies the truth of P_{k+1} , where P_{k+1} is

$$133 \mid (11^{k+1+2} + 12^{(2k+2)+1})$$

$$\begin{aligned} \text{Now, } 11^{k+1+2} + 12^{(2k+2)+1} &= 11^{k+2} 11^2 + 12^{2k+1} 12^2 \\ &= 11^{k+2} 11 + (133m - 11^{2k+1}) 12^2 \quad [\text{by induction assumption}] \\ &= 11^{k+2} 11 + (133m)(144) - (11^{k+2})(144) \\ &= 133(144m) - 133(11^{k+2}) \\ &= 133(144m - 11^{k+2}) \end{aligned}$$

Thus, $133 \mid (11^{k+1+2} + 12^{2(k+1)+1})$

This shows that the result holds for $n = k + 1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so P_n is true for each $n \in \mathbf{N}$.

Example 11 : Show that $14 \mid (3^{4n-2} + 5^{2n-1})$ for all natural number n .

Solution : Let P_n denote the statement $14 \mid (3^{4n-2} + 5^{2n-1})$.

For $n = 1$, we have

$$3^{4n-2} + 5^{2n-1} = 3^2 + 5 = 14 \text{ which is divisible by } 14.$$

Assume that P_n is true for some natural number n , say k . That is, assume that $14 \mid (3^{4k-2} + 5^{2k-1})$ is true for some natural number k . Suppose $3^{4k-2} + 5^{2k-1} = 14m$ for some natural number m . We now show that the truth of P_k implies the truth of P_{k+1} , that is, we show that $14 \mid [(3^{4(k+1)-2} + 5^{2(k+1)-1})]$.

We have

$$\begin{aligned} &= 3^{4(k+1)-2} + 5^{2(k+1)-1} = 3^{4k-2} \cdot 3^4 + 5^{2k-1} \cdot 5^2 \\ &= (14m - 5^{2k-1}) \cdot 3^4 + 5^{2k-1} \cdot 5^2 \quad [3^{4k-2} = 14m - 5^{2k-1}] \\ &= (14m - 81) + 5^{2k-1}(-81 + 25) \\ &= (14m)(81) - (5^{2k-1})(56) = 14[81m - 4 \cdot 5^{2k-1}] \\ \Rightarrow &14 \mid [3^{4(k+1)-2} + 5^{2(k+1)-1}] \end{aligned}$$

This shows that the result holds for $n = k+1$; therefore, the truth of P_k implies the truth of P_{k+1} . The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number n .

Check Your Progress – 1

Use the principle of mathematical induction to prove the following formulae.

1. $1 + 3 + 5 + \dots + (2n-1) = n^2 \quad \forall n \in \mathbf{N}$
2. $1^2 + 2^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1) \quad \forall n \in \mathbf{N}$
3. $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \forall n \in \mathbf{N}$
4. $1(2^2) + 2(3^2) + \dots + n(n+1)^2 \quad \forall n \in \mathbf{N}$
5. $8 \mid (3^n - 1) \quad \forall n \in \mathbf{N}$
6. $24 \mid (5^{2n} - 1) \quad \forall n \in \mathbf{N}$
7. $1 + \frac{1}{2} + \dots + \frac{1}{2^n} < 2. \quad \forall n \in \mathbf{N}$
8. $1 + 2 + \dots + n < (2n+1)2 \quad \forall n \in \mathbf{N}$

Check Your Progress – 1

1. Let P_n denote the statement

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

For $n = 1$, P_n becomes $1 = 1^2$ or $1=1$ which is clearly true.

Assume that P_k is true for $k \in \mathbb{N}$

That is, assume that

$$1 + 3 + 5 + \dots + (2k - 1) = k^2$$

For $n = k + 1$, we have

$$P_{k+1} : 1 + 3 + \dots + (2k - 1) + (2k + 1) = (k + 1)^2$$

Now,

$$\begin{aligned} &1 + 3 + \dots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) \quad \text{[by induction assumption]} \\ &= (k + 1)^2 \end{aligned}$$

2. Clearly result is true for $n = 1$. Assume that result holds for $n = k$, that is,

$$1^2 + 2^2 + \dots + k^2 = \frac{1}{6} k (k + 1) (2k + 1)$$

For $n = k + 1$,

$$\text{LHS} = 1^2 + 2^2 + \dots + k^2 + (k + 1)^2$$

$$= \frac{1}{6} k (k + 1) (2k + 1) + (k + 1)^2$$

$$= \frac{1}{6} (k + 1) [k (2k + 1) + 6(k + 1)]$$

$$= \frac{1}{6} (k + 1) [2k^2 + 7k + 6]$$

$$= \frac{1}{6} (k + 1) [2k^2 + 3k + 4k + 6]$$

$$= \frac{1}{6} (k + 1) [k(2k + 3) + 2(2k + 3)]$$

$$= \frac{1}{6} (k + 1)(k + 2)(2k + 3)$$

$$= \frac{1}{6} (k + 1)(k + 1 + 1)(2k + 1 + 1)$$

The result holds for $n = k + 1$.

3. Result holds for $n = 1$.

Assume that

$$\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

For $n = k + 1$,

$$\begin{aligned} \text{LHS} &= \frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{1}{(2k+1)(2k+3)} [k(2k+3) + 1] \\ &= \frac{1}{(2k+1)(2k+3)} [2k^2 + 2k + k + 1] \\ &= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} = \frac{(k+1)}{(2k+3)} = \frac{(k+1)}{2(k+1)+1} \end{aligned}$$

Result holds for $n = k + 1$.

4. The result holds for $n = 1$.

Assume that

$$1(2^2) + 2(3^2) + \dots + k + (k+1)^2 = \frac{1}{12} k(k+1)(k+2)(3k+5)$$

For $n = k + 1$.

$$\begin{aligned} \text{LHS} &= (1)(2)^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2 \\ &= \frac{1}{12} k(k+1)(k+2)(3k+5) + (k+1)(k+1)^2 \\ &= \frac{1}{12} k(k+1)(k+2) [k(3k+5) + 12(k+2)] \\ &= \frac{1}{12} k(k+1)(k+2) [3k^2 + 17k + 24] \\ &= \frac{1}{12} k(k+1)(k+2) [3k^2 + 8k + 9k + 24] \\ &= \frac{1}{12} k(k+1)(k+2) [k(3k+8) + 3(k+8)] \\ &= \frac{1}{12} k(k+1)(k+2)(k+3)(3k+8) \\ &= \frac{1}{12} k(k+1)(k+1+1)(k+1+2)[(3k+1)+5] \end{aligned}$$

The result holds for $n = k+1$.

5. The result holds for $n = 1$.

Assume that $8 \mid (3^{2k} - 1)$ for same $k \in \mathbf{N}$.

Let $3^{2k} - 1 = 8m$ for some $m \in \mathbf{N}$.

$$\begin{aligned}\text{Next, } 3^{2(k+1)} - 1 &= 3^{2k} \cdot 3^2 - 1 = (8m + 1)(9) - 1 \\ &= 72m + 9 - 1 = 8(9m + 1)\end{aligned}$$

This shows that $8 \mid (3^{2(k+1)} - 1)$

The result holds for $n = k+1$.

6. The result holds for $n = 1$

Assume $24 \mid (5^{2k} - 1)$ for same $k \in \mathbf{N}$.

$$\Rightarrow 5^{2k} - 1 = 24m \text{ for same } m \in \mathbf{N}.$$

For $n = k+1$,

$$\begin{aligned}5^{2k+2} - 1 &= 5^{2k} \cdot 5^2 - 1 \\ &= (24m + 1)(25) - 1 \\ &= (24)(25)m + 25 - 1 \\ &= 24(25m + 1)\end{aligned}$$

Thus, $24 \mid (5^{2k+2} - 1)$

The result holds for $n = k+1$.

7. The result holds for $n = 1$, as $1 + \frac{1}{2} < 2$

Assume that

$$1 + \frac{1}{2} + \dots + \frac{1}{2^k} < 2 \text{ for same } k \in \mathbf{N}.$$

$$\Rightarrow \frac{1}{2} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^k} \right) < 1$$

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{k+1}} < 1 + 1 = 2.$$

The result holds for $n = 1$.

8. The result holds for $n = 1$.

Assume that the result holds for $n = k$, that is ,

$$1 + 2 + \dots + k < (2k + 1)^2$$

We have,

$$\begin{aligned}1 + 2 + \dots + k + (k+1) \\ &< (2k + 1)^2 + (k + 1) \\ &= 4k^2 + 4k + 1 + k + 1 \\ &< 4k^2 + 12k + 9 \quad [\because 7k + 7 > 0] \\ &= (2k + 3)^2\end{aligned}$$

Thus, the result holds for $n = k + 1$.

4.4 SUMMARY

The unit is for the purpose of explaining the Principle of Mathematical Induction, one of the very useful mathematical tools. A large number of examples are given to explain the applications of the principle.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 4.3**.