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## UNIT 2    COMPLEX NUMBERS

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### Structure

- 2.0    Introduction
- 2.1    Objectives
- 2.2    Complex Numbers
- 2.3    Algebra of Complex Numbers
- 2.4    Conjugate and Modules of a Complex Number
- 2.5    Representation of a Complex Numbers as Points in a Plane and Polar form of a Complex Number
- 2.6    Powers of Complex Numbers
- 2.7    Answers to Check Your Progress
- 2.8    Summary

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### 2.0    INTRODUCTION

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All the numbers with which we have dealt so far were real numbers. However, some solutions in mathematics, such as solving quadratic equations require a new set of numbers. This new set of numbers is called the set of **complex numbers**.

If we solve the equation  $x^2 = 4$  for  $x$ , we find the equation has two solutions.

$$x^2 = 4 \Rightarrow x = \sqrt{4} = 2 \text{ or } x = -\sqrt{4} = -2.$$

If we solve the equation  $x^2 = -1$  in a similar way, we would expect it to have two solutions also.

$$x^2 = -1 \text{ should imply } x = \sqrt{-1} \text{ or } x = -\sqrt{-1}.$$

Each proposed solution of the equation  $x^2 = -1$  involves the symbol  $\sqrt{-1}$ . For years it was believed that square roots of negative numbers denoted by  $\sqrt{-5}$ ,  $\sqrt{-2}$  and  $\sqrt{-6}$  were nonsense. In the 17<sup>th</sup> century, these symbols were termed *imaginary numbers* by Rene Descartes (1596-1650). Now, the imaginary numbers are no longer thought to be impossible. In fact imaginary numbers have important uses in several branches mathematics and physics.

The number  $\sqrt{-1}$  occurs so often in mathematics, that we give it a special symbol. We use better  $i$  to denote  $\sqrt{-1}$ . Since  $i$  stand for  $\sqrt{-1}$ , it immediately follows that  $i^2 = -1$ . The power of  $i$  with natural exponent produces an interesting pattern, as follows :

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad i^7 = -i, \quad i^8 = 1$$

$$\text{also } i^{-1} = -i, \quad i^{-2} = -1, \quad i^{-3} = i, \quad i^{-4} = 1$$

## 2.1 OBJECTIVES

After studying this unit, you will be able to :

- define complex number and perform algebraic operations such as addition, subtraction, multiplication and division on the complex numbers;
- find modulus, argument and conjugate of a complex number;
- represent complex numbers in the argand plane;
- write polar form of a complex number;
- use Demoivre's theorem; and
- find cube roots of unity and verify some of the identities involving them.

## 2.2 COMPLEX NUMBERS

**Definition :** A *complex number* is any number that can be put in the form  $a + bi$ , where  $a$  and  $b$  are real number and  $i = \sqrt{-1}$ . The form  $a + bi$  is called **standard form** for complex number. The number  $a$  is called the real part of the complex number. The number  $b$  is called imaginary part of the complex number.

We usually denote a complex number by  $z$ . We write  $z = a + bi$ . The real part of  $z$  is denoted by  $\text{Re}(z)$  and the imaginary part of  $z$  is denoted by  $\text{Im}(z)$ .

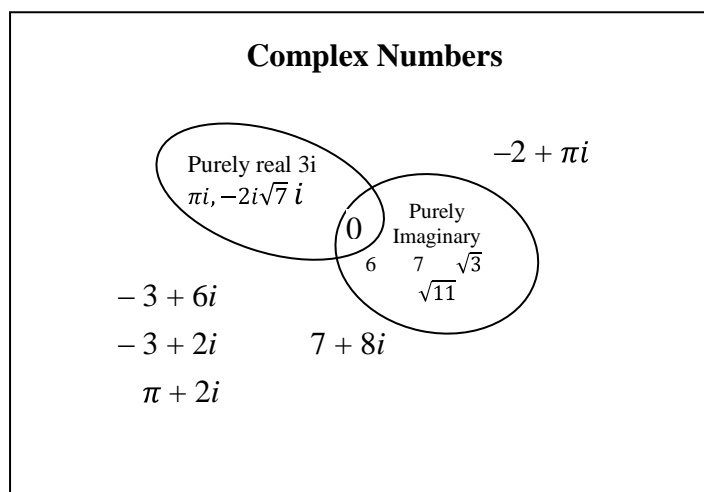


Figure 1

If  $b = 0$ , the complex number  $a + bi$  is the real number  $a$ . Thus, any real number is a complex number with zero imaginary part. In other words, the set of real numbers is a subset of the set of complex numbers.

### Equality of two Complex Numbers

Two complex numbers are equal if and only if their real parts are equal and also their imaginary parts are equal.

Thus if,  $z_1 = a + bi$  and  $z_2 = c + di$  are two complex numbers, then  $z_1 = z_2$ , that is,  $a + bi = c + di$  if and only if  $a = c$  and  $b = d$ .

**Example 1** (a) Find  $x$  and  $y$  if  $3x + 4i = 12 - 8yi$   
 (b) Find  $a$  and  $b$  if  $(4a - 3) + 7i = 5 + (2b - 1)i$

**Solution :**

(a) Since the two complex numbers are equal, their real parts are equal and their imaginary parts are equal :

$$3x = 12 \text{ and } 4 = -8y \Rightarrow x = 4 \text{ and } y = -1/2$$

(b) The real parts are  $4a - 3$  and  $5$ . The imaginary parts are  $7$  and  $2b - 1$ .

$$4a - 3 = 5 \text{ and } 7 = 2b - 1 \Rightarrow 4a = 8 \text{ and } 2b = 8 \Rightarrow a = 2 \text{ and } b = 4.$$

## 2.3 ALGEBRA OF COMPLEX NUMBERS

### Addition of two Complex Numbers

Two complex numbers such as  $z_1 = a + bi$  and  $z_2 = c + di$  are added as if they are algebraic binomials:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Observe that  $a + bi = (a + 0i) + (0 + bi)$ . In other words,  $a + bi$  is the sum of the real number  $a$  and the imaginary number  $bi$ .

Also observe that  $z_1 + z_2$  is a complex number.

### Illustration

$$(i) \quad (3 + 4i) + (7 - 6i) = (3 + 7) + (4 - 6)i = 10 - 2i$$

$$(ii) \quad (8 - 3i) + (6 - 2i) = (8 + 6) + (-3 - 2)i = 14 - 5i$$

### Subtraction of Complex Numbers

If  $z_1 = a + bi$  and  $z_2 = c + di$ , we define  $z_1 - z_2$  as  $z_1 + (-z_2)$ .

$$\text{That is, } z_1 - z_2 = (a + bi) + ((-c) + (-d)i) = (a - c) + (b - d)i$$

**Example 2**

Fill in the blanks

- (i)  $(-4 + 10i) + (-1 + 2i) = \dots$  (ii)  $(-6 + 17i) + (4 - 11i) = \dots$   
 (iii)  $(-4 + 2i) + (7 - 2i) = \dots$  (iv)  $(3 - 5i) + (-3 + 5i) = \dots$

**Solution**

- (i)  $-5 + 12i$  (ii)  $-2 + 6i$   
 (iii)  $3$  (iv)  $0$

**Example 3**

Fill in the blanks

- (i)  $-(3 + 4i) = \dots$  (ii)  $(3 - 2i) - (4 - 3i) = \dots$   
 (iii)  $(2 + 3i) - (i) = \dots$  (iv)  $(5 + 2i) - 2 = \dots$

**Solution**

- (i)  $-3 - 4i$  (ii)  $-1 + i$   
 (iii)  $2 + 2i$  (iv)  $3 + 2i$

**Multiplication of Complex Numbers**

Two complex numbers such as  $z_1 = a + bi$  and  $z_2 = c + di$  are multiplied as if they were algebraic binomials, with  $i^2 = -1$ ;

$$z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

By definition, product of two complex numbers is again a complex number. Also observe that  $yi = (y + 0i)(0 + 1i)$  and is, thus the product of the real number  $y$  and the imaginary number  $i$ .

**Illustration 1**

$$(3 + 2i)(4 + 5i) = 12 + 5i + 8i + 10i^2 = 12 + 15i + 8i - 10 = 2 + 23i \quad [\because i^2 = -1]$$

$$\text{and } (2 + 5i)(7 + 3i) = 14 + 6i + 35i + 15i^2 = 14 + 6i + 35i - 15 = -1 + 41i$$

$$[\because i^2 = -1]$$

**Example 4** Perform the indicated operations and write the results in the form of  $a + bi$

- (i)  $(2 + 3i)^2$  (ii)  $(1 + i)^3$   
 (iii)  $(\sqrt{5} + 7i)(\sqrt{5} - 7i)$

**Solution**

$$(i) \quad (2 + 3i)^2 = (2 + 3i)(2 + 3i) = (2)(2) + (2)(3i) + (2)(3i) + (3i)(3i)$$

$$= 4 + 6i + 6i + 9i^2 = 4 + 12i - 9 = -5 + 12i$$

$$(ii) \quad (1 + i)^3 = (1 + i)(1 + i)(1 + i) = (1 + i + i + i^2)(1 + i) = (1 + i + i - 1)(1 + i)$$

$$= 2i(1 + i) = 2i - 2i^2 = -2 + 2i$$

$$(iii) \quad (\sqrt{5} + 7i)(\sqrt{5} - 7i) = (\sqrt{5})(\sqrt{5}) - (\sqrt{5})(7i) + (\sqrt{5})(7i) - (7i)(7i)$$

$$= 5 + 7(\sqrt{5}i) - 7(\sqrt{5}i) - 49i^2 = 5 + 49 = 54$$

**Multiplicative Inverse of a Non-Zero Complex Number**

If  $a + ib \neq 0$  is any complex number, then there exists a complex number  $x + iy$  such that

$$(a + ib)(x + iy) = 1 + 0i = \text{the multiplicative identity in } C.$$

The number  $x + iy$  is called the multiplicative inverse of  $(a + ib)$  in  $C$ .

$$\text{Now, } (a + ib)(x + iy) = 1 + 0i \Rightarrow (ax - by) + i(ay - bx) = 1 + 0i$$

[multiplication of complex numbers]

$$\Rightarrow ax - by = 1 \text{ and } ay + bx = 0 \quad [\text{equality of two complex numbers}]$$

$$\Rightarrow ax - by - 1 = 0 \text{ and } ay + bx = 0$$

Solving these equations for  $x$  and  $y$ , we have

$$x = \frac{a}{a^2 + b^2} \quad (1)$$

$$y = \frac{-b}{a^2 + b^2} \quad (2)$$

both of which exist in  $\mathbf{R}$ , because  $(a + ib) \neq 0$  i.e., at least one of  $a, b$  is different from zero.

Thus, the multiplicative inverse is of  $a + ib$  is

$$x + iy = \frac{a}{a^2 + b^2} - i \frac{a}{a^2 + b^2} = \frac{a - ib}{a^2 + b^2}$$

Thus, every non-zero complex number has a multiplicative inverse in  $\mathbb{C}$ .

### Division in Complex Numbers

If  $Z_1 = x + iy$  and  $Z_2 = a + ib \neq 0$ ,

then

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{x + iy}{a + ib} = (x + iy) \frac{1}{(a + ib)} \\ &= (x + iy) \frac{(a - ib)}{(a^2 + b^2)} \\ &= \frac{ax + by}{a^2 + b^2} + i \frac{bx - ay}{a^2 + b^2} \end{aligned}$$

**Example 5** If  $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$ , then show that  $a = 1$  and  $b = 0$

**Solution:** We have

$$\begin{aligned} \frac{1-i}{1+i} &= \frac{(1-i)(1-i)}{(1-i)(1+i)} = \frac{(1-i)^2}{1^2 - i^2} \\ &= \frac{1 - 2i + i^2}{2} = \frac{1 - 2i - 1}{2} = -i \end{aligned}$$

$$\text{Thus, } \left(\frac{1-i}{1+i}\right)^{100} = (-i)^{100} = 1$$

$$\therefore a + ib = 1 \Rightarrow a = 1 \text{ and } b = 0$$

**Example 6 :** If  $x = -2 - \sqrt{3}i$ , find the value of  $2x^4 + 5x^3 + 7x^2 - x + 41$ .

**Solution :**

$$\begin{aligned} x = -2 - \sqrt{3}i &\Rightarrow x + 2 = -\sqrt{3}i \Rightarrow (x + 2)^2 = (-\sqrt{3}i)^2 \\ \Rightarrow x^2 + 4x + 4 &= -3 \text{ or } x^2 + 4x + 7 = 0 \end{aligned}$$

We now divide  $2x^4 + 5x^3 + 7x^2 - x + 41$  by  $x^2 + 4x + 7$

$$\begin{array}{r}
 x^2 + 4x + 7 \overline{) 2x^4 + 5x^3 + 7x^2 - x + 41} \quad 2x^2 - 3x + 5 \\
 \underline{2x^4 + 8x^3 + 14x^2} \phantom{-x + 41} \\
 -3x^3 - 7x^2 - x + 41 \\
 \underline{-3x^3 - 12x^2 - 21x} \phantom{+ 41} \\
 5x^2 + 20x + 41 \\
 \underline{5x^2 + 20x + 35} \\
 6
 \end{array}$$

Thus,  $2x^4 + 5x^3 + 7x^2 - x + 41 = (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$

$$= (0)(2x^2 - 3x + 5) + 6 = 6$$

$\therefore$  value of  $2x^4 + 5x^3 + 7x^2 - x + 41$  for  $x = -2 - \sqrt{3}i$  is 6.

### Check Your Progress - 1

1. Is the following computation correct ?

$$\sqrt{-5} \sqrt{-7} = \sqrt{(-5)(-7)} = \sqrt{35}$$

2. Express each one of the following in the standard form  $a + ib$ .

$$(i) \quad \frac{1}{5-4i} \quad (ii) \quad \frac{7+2i}{2-7i} \quad (iii) \quad \frac{1}{\cos \theta + i \sin \theta} \quad (iv) \quad \frac{2-\sqrt{-25}}{1-\sqrt{-16}}$$

3. Find the multiplicative inverse of

$$(i) \quad \frac{1+i}{1-i} \quad (ii) \quad (1 + \sqrt{3}i)^2 \quad (iii) \quad (1+i)(1+2i)$$

4. Find the value of  $x^4 - 4x^3 + 4x^2 + 8x + 40$

when  $x = 3 + 2i$ .

5. If  $(x + iy)^{1/3} = a + ib$ , prove that

$$\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$$

6. Find the smallest positive integer for which

$$\left( \frac{1+i}{1-i} \right)^n = 1$$

## 2.4 CONJUGATE AND MODULUS OF A COMPLEX NUMBER

### Conjugate of a Complex Number

**Definition :** If  $z = x + i y$ ,  $x, y \in \mathbf{R}$  is a complex number, then the complex number  $x - i y$  is called conjugate of  $z$  and is denoted by  $\bar{z}$ .

For instance,

$$\overline{2+3i} = 2-3i, \quad \overline{3-4i} = 3+4i, \quad \bar{i} = -i \text{ and}$$

$$\bar{3} = \overline{3+0i} = 3-0i = 3$$

### Some properties of Complex Conjugates

1.  $\bar{\bar{z}} = z$
2.  $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
3.  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
4.  $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$  if  $z_2 \neq 0$
5. If  $z = a + ib$ , then
 
$$z + \bar{z} = 2a = 2(\operatorname{Re}(z))$$
 and  $z - \bar{z} = 2ib = 2i \operatorname{Im}(z)$
6.  $z = \bar{z} \Leftrightarrow z$  is real
7.  $z = -\bar{z} \Leftrightarrow z$  is imaginary

### Modulus of a Complex Number

**Definition :** If  $z = x + iy$ ,  $x, y \in \mathbf{R}$  is a complex number, then the real number  $\sqrt{x^2 + y^2}$  is called the modulus of the complex number  $z$ , and is denoted by  $|z|$ .

For instance, if  $z = 2 + 3i$ , then  $|z| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

and if  $z = 5 - 12i$ , then  $|z| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

Note that

$$|z| = |-z| = |-\bar{z}| = |z|.$$

and if  $c$  is a real number, then  $|cz| = |c| |z|$

### Some properties of Modulus of complex numbers

1.  $|z|^2 = z \bar{z}$
2.  $|z| = 0 \Leftrightarrow z = 0$



$$3. \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2} \text{ if } z \neq 0 \quad 4. \quad |z_1 z_2| = |z_1| |z_2|$$

$$5. \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \text{ if } z_2 \neq 0 \quad 6. \quad -|z| \leq z \leq |z|$$

$$7. \quad |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2 \\ = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2)$$

**Example 7:** If  $a + ib \neq 0$ , show that

$$\left| \frac{a - ib}{a + ib} \right| = 1$$

**Solution :** Let  $-z = a + ib$ , then  $z = a - ib$

Since  $|-z| = |\bar{z}|$ , we get

$$1 = \frac{|\bar{z}|}{|z|} = \left| \frac{\bar{z}}{z} \right| = \left| \frac{a-ib}{a+ib} \right| \quad \left[ \because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right]$$

**Example 8:** If  $x + iy = \sqrt{\frac{a+ib}{c+id}}$ , then  $x^2 + y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

**Solution :**  $(x + iy)^2 = \frac{a+ib}{c+id}$

$$\Rightarrow |(x + iy)^2| = \left| \frac{a+ib}{c+id} \right|$$

$$\Rightarrow |x + iy|^2 = \left| \frac{a+ib}{c+id} \right| \quad \left[ \because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right]$$

$$\Rightarrow (\sqrt{x^2 + y^2})^2 = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

$$\Rightarrow x^2 + y^2 = \sqrt{\frac{a^2 + b^2}{c^2 + d^2}}$$

**Example 9:** If  $(a - ib)(x + iy) = (a^2 + b^2)i$  and  $a + ib \neq 0$ , show that  $x = b$  and  $y = a$ .

**Solution:** Let  $z = a + ib$ , then  $\bar{z} = a - ib$

$$\text{Now, } (a + ib)(x - iy) = (a^2 + b^2)i$$

$$\Rightarrow \bar{z}(x + iy) = z \bar{z} i$$

$$\Rightarrow x + iy = \bar{z} i = (a - ib)i = ai + b$$

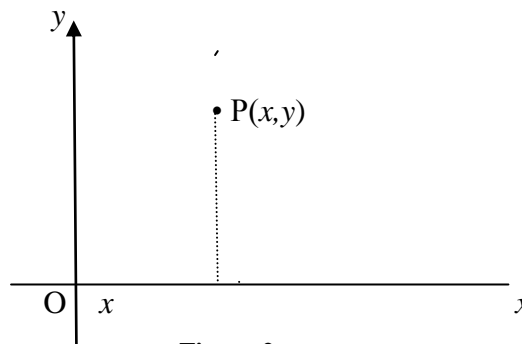
$$\Rightarrow x = b, \quad y = a \quad [\text{by definition of equality of Complex Numbers}]$$

**Check Your Progress – 2**

1. Let  $Z = x + iy$  and  $\omega = \frac{1 - iZ}{Z - i}$ . If  $|\omega| = 1$ , show that  $Z$  is purely real.
2. If  $|Z| = 1$ ,  $Z \neq -1$  show that  $\frac{Z - 1}{Z + 1}$  is purely imaginary
3. If  $|Z - i| = |Z + i|$ , show that  $\text{Im}(Z) = 0$ .
4. If  $(a + bi)(3 + i) = (1 + i)(2 + i)$ , find  $a$  and  $b$ .
5. If  $(\cos \theta + i \sin \theta)^2 = x + iy$ , that show  $x^2 + y^2 = 1$ .

## 2.5 REPRESENTATION OF A COMPLEX NUMBERS AS POINTS IN A PLANE AND POLAR FORM OF A COMPLEX NUMBER

Let  $OX$  and  $OY$  be two rectangular axes in a plane with their point of intersection as the origin.



**Figure 2**

To each ordered pair  $(x, y)$  there corresponds a point  $P$  in the plane such that the  $x$ -coordinate of  $P$  is  $x$  and the  $y$ -coordinate of  $P$  is  $y$ . Thus, to a complex number  $z = x + iy$  where corresponds a point  $P(x, y)$  in the plane. Conversely, to every point  $P(x', y')$  there corresponds a complex number  $x' + iy'$ .

Thus, there is one-to-one correspondence between the set  $C$  of all complex numbers and the set of all the points in a plane.

For Example, the complex number  $4 + 3i$  is represented by the point  $(4, 3)$  and the point  $(-3, -4)$  represents the complex number  $-3 - 4i$ .

We note that the points corresponding to the complex numbers of the type  $a$  lie on the  $x$ -axis and the complex numbers of the type  $bi$  are represented by points on the  $y$ -axis.

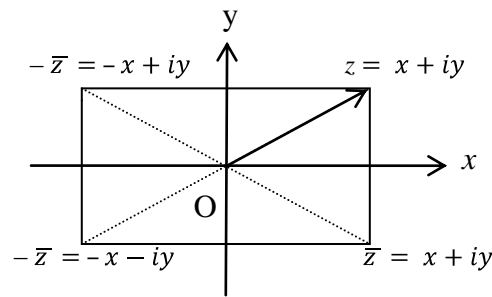


Figure 3

Note that the points  $z$  and  $-z$  are symmetric with respect to point  $O$ , while points  $z$  and  $\bar{z}$  are symmetric with respect to the real axis, since if  $z = x + iy$ , then  $-z = (-x) + i(-y)$  and  $\bar{z} = x + i(-y)$ . See Figure 3.

**Remark :** Since the points on the  $x$ -axis represent complex number  $z$  with  $I(z) = 0$ , the  $x$ -axis is also known as the real axis. Points on the  $y$ -axis represent complex numbers  $z$  with  $R(z) = 0$ , the  $y$ -axis is also known as the imaginary axis. The plane is called as the *Argand plane*, *Argand diagram*, *complex plane* or *Gaussian plane*.

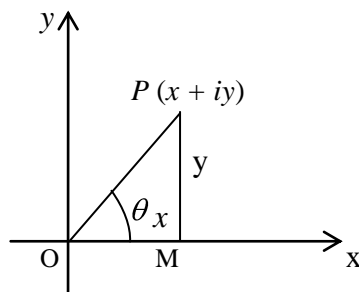


Figure 4

Note that  $OP = \sqrt{x^2 + y^2} = |z|$

### Polar Representation of Complex Numbers

Let  $P(z)$  represents the complex number  $z = x + iy$  as shown in the complex plane. Recall that the modulus or the absolute value of the complex number  $z$  is defined as the length  $OP$ . It is denoted by  $|z|$ . Thus if  $r = OP$ ; we have

$$r = |z| = OP$$

$$= \sqrt{OM^2 + PM^2} = \sqrt{x^2 + y^2}$$

$$= \sqrt{[Re(z)]^2 + [Im(z)]^2}$$

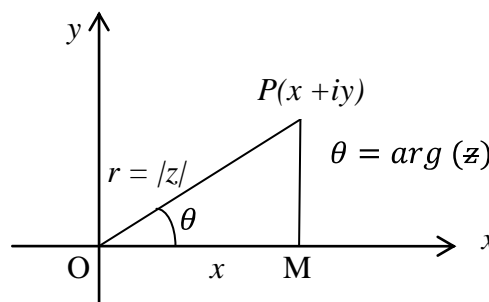


Figure 5

If  $\theta$  be the angle which  $OP$  makes with  $OX$  in anticlockwise sense, then  $\theta$  is called the *argument* or *amplitude* of the complex number  $z = x + iy$ .

Now in the right triangle  $\triangle OMP$ ,

$$x = OM = OP \cos \theta = r \cos \theta \quad (1)$$

$$y = MP = OP \sin \theta = r \sin \theta \quad (2)$$

Thus, the complex number  $z$  can be written as

$$z = x + iy = r \cos \theta + ir \sin \theta = r (\cos \theta + i \sin \theta)$$

This, is known as the *polar* form of the complex number.

Squaring and adding (1) and (2) we have

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2 \cdot 1 = r^2$$

[Pythagorean identity]

$$\text{Thus } r^2 = x^2 + y^2 \text{ or } r = \sqrt{x^2 + y^2}$$

which is the *modulus* of the complex number  $z = x + iy$ .

Dividing (2) and (1), we have

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta \Rightarrow \tan \theta = \frac{y}{x}.$$

$\theta$  is the argument of the complex number  $z = x + iy$ .

The value of  $\theta$  ( $-\pi < \theta \leq \pi$ ) is called the *principal* value of the argument or amplitude of  $z$ . We denote it by  $\text{Arg } z$  instead of  $\arg z$ .

## 2.6 POWERS OF COMPLEX NUMBERS

### Product of $n$ Complex Numbers

We first take up product of complex numbers.

$$\text{If } z_1 = r_1 (\cos \theta_1 + i \sin \theta_1), \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2), \dots \dots \dots$$

$$z_n = r_n (\cos \theta_n + i \sin \theta_n), \text{ then}$$

$$z_1 z_2 \dots z_n = r_1 r_2 \dots r_n [\cos (\theta_1 + \theta_2 + \dots + \theta_n) + i \sin (\theta_1 + \theta_2 + \dots + \theta_n)]$$

However, we shall not prove this statement.

When  $r_1 = r_2 = \dots \dots \dots r_n = 1$ , we get

$$(\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2) \dots \dots \dots (\cos \theta_n + i \sin \theta_n)$$

$$= \cos (\theta_1 + \theta_2 + \dots \dots \dots \theta_n) + i \sin (\theta_1 + \theta_2 + \dots \dots \dots \theta_n) \quad (1)$$

**Corollary 1.**  $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$  and

$$2. \quad \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

**Proof** From (1), above we have

$$\begin{aligned} & \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \\ &= (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2) \\ &= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \end{aligned}$$

Equating real and imaginary parts, we get

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \\ \text{and } \sin(\theta_1 + \theta_2) &= \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 \end{aligned}$$

### De Moivre's Theorem (for Integral Index)

Taking  $\theta_1 = \theta_2 = \dots = \theta_n = \theta$  in (1) we obtain

$$(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$$

This proves the result for positive integral index.

However, it is valid for every integer  $n$ .

**Example 10 :** Use De Moivre's theorem to find  $(\sqrt{3} + i)^3$ .

**Solution :** We first put  $\sqrt{3} + i$  in the polar form.

$$\text{Let } \sqrt{3} + i = r(\cos\theta + i \sin\theta)$$

$$\Rightarrow \sqrt{3} = r \cos\theta \text{ and } 1 = r \sin\theta$$

$$\Rightarrow (\sqrt{3})^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow r^2 = 4 \Rightarrow r = 2$$

$$\text{Thus, } \sqrt{3} + i = 2(\cos\theta + i \sin\theta)$$

$$\Rightarrow \sqrt{3} = 2 \cos\theta \text{ and } 1 = 2 \sin\theta$$

$$\Rightarrow 2 \cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = 30^\circ.$$

$$\text{Now, } (\sqrt{3} + i)^3 = [2\cos(30^\circ) + i \sin(30^\circ)]^3$$

$$= 2^3 [\cos(30^\circ) + i \sin(30^\circ)]^3$$

$$= 8 [\cos(3 \times 30^\circ) + i \sin (3 \times 30^\circ)] \text{ [De Moivre's theorem]}$$

$$= 8 (\cos 90^\circ + i \sin 90^\circ) = 8(0 + i)$$

$$= 8i$$

### Cube Roots of Unity

$$\text{Let } x = (1)^{1/3}$$

$$\Rightarrow x^3 = 1 \Rightarrow x^3 - 1 = 0 \Rightarrow (x - 1)(x^2 + x + 1) = 0$$

$$\text{Therefore, either } x - 1 = 0 \Rightarrow (x - 1)(x^2 + x + 1) = 0$$

$$\Rightarrow \text{either } x = 1 \text{ or } x = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm \sqrt{(-3)}}{2} = \frac{-1 \pm i\sqrt{(3)}}{2}$$

$$\text{Thus, the three cube roots of unity are, } 1, \frac{-1}{2} + i \frac{\sqrt{3}}{2}, \frac{-1}{2} - \frac{i\sqrt{3}}{2}$$

Hence, there are three cube roots of unity.

Out of these one root (i.e., 1) is real and remaining two viz.,

$$\frac{-1 + i\sqrt{(3)}}{2} \text{ and } \frac{-1 - i\sqrt{(3)}}{2} \text{ are complex.}$$

We usually denote the cube root  $\frac{-1}{2} + \frac{\sqrt{3}}{2} i$  by  $\omega$  note that

$$\omega^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2 = \frac{1}{4} - \frac{3}{4} - \frac{2\sqrt{3}}{4} i = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

Hence, the cube roots of unity are 1,  $\omega$ ,  $\omega^2$ .

Also, note that  $\omega^3 = 1$ .

### Some properties of Cube Roots of Unity

$$1. \quad 1 + \omega + \omega^2 = 0$$

$$2. \quad \omega^3 = 1$$

$$3. \quad \frac{1}{\omega} = \omega^2 \text{ and } \frac{1}{\omega^2} = \omega$$

**Example 11:** If 1,  $\omega$ ,  $\omega^2$  are cube roots of unity, show that

- (i)  $(1 + \omega)^2 - (1 + \omega)^3 + \omega^2 = 0$   
 (ii)  $(2 - \omega)(2 - \omega^2)(2 - \omega^{10})(2 - \omega^{11}) = 49$

**Solution :** (i) As  $1 + \omega + \omega^2 = 0$ , we get

$$1 + \omega = -\omega^2 \quad \text{and} \quad 1 + \omega^2 = -\omega$$

Thus,

$$\begin{aligned} & (1 + \omega)^2 - (1 + \omega^2)^3 + \omega^2 \\ &= (-\omega^2)^2 - (-\omega)^3 + \omega^2 \\ &= \omega^4 + \omega^3 + \omega^2 = \omega^3\omega + 1 + \omega^2 \\ &= \omega + 1 + \omega^2 = 0 \end{aligned}$$

(ii) Since  $\omega^{10} = (\omega^3)^3 \omega = \omega$

and  $\omega^{11} = (\omega^3)^3 \omega^2 = \omega^2$ ,

$$\begin{aligned} & \text{Thus } (2 - \omega)(2 - \omega^2)(2 - \omega^{10})(2 - \omega^{11}) \\ &= (2 - \omega)(2 - \omega^2)(2 - \omega)(2 - \omega^2) \\ &= [(2 - \omega)(2 - \omega^2)]^2 \\ &= [4 - 2\omega - 2\omega^2 + \omega^3]^2 \\ &= [4 - 2(\omega + \omega^2) + 1]^2 \\ &= [4 - 2(-1) + 1]^2 \quad [\because \omega + \omega^2 = -1] \\ &= 7^2 = 49 \end{aligned}$$

**Example 12:** If  $x = a + b$ ,  $y = a\omega + b\omega^2$

and  $z = a\omega^2 + b\omega$ , show that

$$xyz = a^3 + b^3$$

**Solution:**  $xyz$

$$\begin{aligned} &= (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega) \\ &= (a + b)(a^3\omega^3 + ab\omega^4 + ab\omega^2 + b^2\omega^3) \\ &= (a + b)(a^2 + ab(\omega^3\omega + \omega^2) + b^2) \quad [\because \omega^3 = 1] \\ &= (a + b)[a^2 + ab(-1) + b^2] \\ &= (a + b)(a^2 - ab + b^2) \\ &= a^3 + b^3 \quad [\because a^3 + b^3 = (a + b)(a^2 - ab + b^2)] \end{aligned}$$

## Check Your Progress – 3

1. Calculate

(i)  $(\cos 30^\circ + i \sin 30^\circ)(\cos 60^\circ + i \sin 60^\circ)$

(ii)  $(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)$

(iii)  $(\cos 45^\circ + i \sin 45^\circ)^2$

2. Use identities

$$\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$$

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \text{ to obtain values of}$$

(i)  $\cos(75^\circ)$

(ii)  $\sin 75^\circ$

(iii)  $\cos(90^\circ + \theta)$

(iv)  $\sin(90^\circ + \theta)$

(v)  $\cos(105^\circ)$

(vi)  $\sin(105^\circ)$

3. Using the identities in Question 2, show that

$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$$

4. If  $1, \omega, \omega^2$  are three cube roots of unity, show that

(i)  $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^6)(1 + \omega^8) = 2$

(ii)  $(1 - \omega^2 + \omega^2)^5 + (1 - \omega^2 - \omega^2)^5 = 32$

(iii)  $(2 + 3\omega + 2\omega^2)^9 = (2 + 3\omega + 3\omega^2)^9 = 1$

5. If  $x = a + b, y = a\omega + b\omega^2$  and

$$z = a\omega^2 + b\omega, \text{ show that}$$

(i)  $x + y + z = 0$

(ii)  $x^2 + y^2 + z^2 = 6ab$

(ii)  $x^3 + y^3 + z^3 = 3(a^3 + b^3)$

**2.7 ANSWERS TO CHECK YOUR PROGRESS**

1. No.

$$\text{The formula } \sqrt{a}\sqrt{b} = \sqrt{ab}$$

holds when at least one of  $a, b \geq 0$ .

2. (i) 
$$\frac{1}{5 - 4i} = \frac{5 + 4i}{(5 - 4i)(5 + 4i)} = \frac{5 + 4i}{25 + 16}$$

$$= \frac{5}{41} + \frac{4}{41}i$$

(ii) 
$$\frac{7 + 2i}{2 - 7i} = \frac{7 + 2i}{-2i^2 - 7i} = \frac{7 + 2i}{(-i)(7 + 2i)} = \frac{-1}{i} = \frac{i^2}{i} = i$$



$$\begin{aligned}
 \text{(iii)} \quad \frac{1}{\cos \theta + i \sin \theta} &= \frac{\cos \theta - i \sin \theta}{(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)} \\
 &= \frac{\cos \theta - i \sin \theta}{(\cos^2 \theta - i^2 \sin^2 \theta)} = \frac{\cos \theta - i \sin \theta}{(\cos^2 \theta + i^2 \sin^2 \theta)} = \cos \theta - i \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \frac{2 - \sqrt{-25}}{1 - \sqrt{-16}} &= \frac{2 - 5i}{1 - 4i} = \frac{(2 - 5i)(1 + 4i)}{(1 - 4i)(1 + 4i)} \\
 &= \frac{2 - 5i + 8i - 20i^2}{1 - 16i^2} \\
 &= \frac{22 + 3i}{17} = \frac{22}{17} + \frac{3}{17}i
 \end{aligned}$$

3. (i) Multiplicative inverse of  $\frac{1+i}{1-i}$  is

$$\frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2 - i^2} = \frac{1 + i^2 - 2i}{1 + 1} = \frac{1 - 1 - 2i}{2} = -i$$

(i) Multiplicative inverse of  $(1 + \sqrt{3}i)^2$  is

$$\begin{aligned}
 \frac{1}{(1 + \sqrt{3}i)^2} &= \frac{(1 - \sqrt{3}i)^2}{((1 + \sqrt{3}i)(1 - \sqrt{3}i))^2} = \frac{1 - 2\sqrt{3}i + 3i^2}{(1 + 3)^2} = \frac{1 - 2\sqrt{3}i - 3}{16} \\
 &= \frac{-2 - 2\sqrt{3}i}{16} = -\frac{1}{8}(1 + \sqrt{3}i)
 \end{aligned}$$

(ii) We have

$$(1 + i)(1 + 2i) = 1 + 1i + 2i + 2i^2 = 1 + 3i - 2 = -1 + 3i$$

Its multiplicative inverse is

$$\begin{aligned}
 \frac{1}{-1 + 3i} &= \frac{-1 - 3i}{(-1 + 3i)(-1 - 3i)} \\
 &= \frac{-1 - 3i}{1 - 9i^2} = \frac{-1 - 3i}{1 + 9} = -\frac{1}{10} - \frac{3}{10}i = -\frac{1}{10}(1 + 3i)
 \end{aligned}$$

$$4. \quad x = 3 + 2i \Rightarrow x - 3 = 2i$$

$$\begin{aligned}
 \Rightarrow (x - 3)^2 &= (2i)^2 \Rightarrow x^2 - 6x + 9 = -4 \\
 \text{or } x^2 - 6x + 13 &= 0
 \end{aligned}$$

Let's divide  $x^4 - 4x^3 + 4x^2 + 8x + 39$  by  $x^2 - 6x + 13$ .

$$\begin{array}{r}
x^2 - 6x + 13 \overline{) x^4 - 4x^3 + 4x^2 + 8x + 40} \quad \left( x^2 + 2x + 3 \right. \\
\underline{\ominus \quad \oplus \quad \ominus} \phantom{+ 40} \\
2x^3 - 9x^2 + 8x + 40 \\
\underline{\ominus \quad \oplus \quad \ominus} \phantom{+ 40} \\
3x^2 - 18x + 40 \\
\underline{\ominus \quad \oplus \quad \ominus} \\
1
\end{array}$$

Thus,  $x^4 - 4x^3 + 4x^2 + 8x + 40$

$$= (x^2 - 6x + 13)(x^2 + 2x + 3) + 1$$

$$= 0 + 1 = 1$$

5.  $x + iy = (a + ib)^3 = a^3 + i^3b^3 + 3a(ib)(a + ib)$

$$= (a^3 - 3a^2b) + i(3a^2b - b^3)$$

$$\Rightarrow x = a^3 - 3a^2b \text{ and } y = 3a^2b - b^3$$

$$\Rightarrow \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = (a^2 - 3b^2) + (3a^2 - b^2) = 4(a^2 - b^2)$$

6. We have  $\frac{1+i}{1-i} = \frac{-i^2+i}{1-i} = \frac{i(1-i)}{1-i} = i$

$$\left( \frac{1+i}{1-i} \right)^n = i^n$$

$\therefore$  The smallest value of  $n$  is 4.

### Check Your Progress – 2

1. Let  $Z = x + iy$

Now,  $|\omega| = 1 \Rightarrow |1 - iZ| = |Z - i|$

$$\Rightarrow |1 - i(x + iy)| = |x + iy - i|$$

$$\Rightarrow |(1 + y) - ix| = |x + (y - 1)i|$$

$$\Rightarrow |(1 + y) - ix|^2 = |x + (y - 1)i|^2$$

$$\Rightarrow (1 + y)^2 + x^2 = x^2 + (y - 1)^2$$

$$\Rightarrow 1 + 2y + y^2 = y^2 - 2y + 1 \Rightarrow 4y = 0 \text{ or } y = 0$$

$\therefore Z = x \Rightarrow Z$  is purely real.

2. Let  $Z = x + iy$

As  $|Z| = 1$ , we get  $x^2 + y^2 = 1$

$$\text{Now, } \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{[(x-1)+iy][(x+1)-iy]}{(x+1)^2 + y^2}$$

$$= \frac{(x^2-1) + y^2 + iy(x+1-x+1)}{x^2 + 2x + 1 + y^2}$$

$$= \frac{(1-1) + 2ixy}{2(x+1)} = \frac{xy}{x+1}i$$

$$\Rightarrow \frac{z-1}{z+1} \text{ is purely imaginary.}$$

3. Let  $z = x + iy$

$$|z-i| = |z+i|$$

$$\Rightarrow |x+iy-i| = |x+iy+i|$$

$$\Rightarrow |x+i(y-1)|^2 = |x+i(y+1)|^2$$

$$\Rightarrow x^2 + (y-1)^2 = x^2 + (y+1)^2$$

$$\Rightarrow (y-1)^2 - (y+1)^2 = 0$$

$$\Rightarrow -4y = 0 \Rightarrow y = 0$$

Thus,  $\text{Im}(z) = 0$

$$4. a + bi = \frac{(1+i)(2+i)}{3+i} = \frac{2-1+3i}{3+i}$$

$$= \frac{1+3i}{3+i} = \frac{(1+3i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{3+3+(9-1)i}{9+1} = \frac{6+8i}{10}$$

$$= \frac{3}{5} + \frac{4}{5}i \Rightarrow a = \frac{3}{5}, \quad b = \frac{4}{5}$$

$$5. |(\cos \theta + i \sin \theta)^2| = |x + iy|$$

$$|\cos \theta + i \sin \theta|^2 = |x + iy|$$

$$\Rightarrow |\cos \theta + i \sin \theta|^2 = \sqrt{x^2 + y^2}$$

$$\Rightarrow \left( \sqrt{\cos^2 \theta + \sin^2 \theta} \right)^2 = \sqrt{x^2 + y^2}$$

$$\Rightarrow x^2 + y^2 = 1$$

**Check Your Progress – 3**

$$1. \text{ (i) } \cos (30^\circ + 60^\circ) + i \sin (30^\circ + 60^\circ)$$

$$= \cos 90^\circ + i \sin 90^\circ = i$$

$$\text{(ii) } (\cos \theta)^2 - i^2 \sin^2 \theta = \sin^2 \theta + \sin^2 \theta = 1$$

$$\text{(iii) } \cos (2(45^\circ)) + i \sin (2(45^\circ))$$

$$= \cos 90^\circ + i \sin 90^\circ = i$$

$$2. \text{ (i) } \cos 75^\circ = \cos (45^\circ + 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$= \frac{(\sqrt{3} - 1)\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\text{(ii) } \sin 75^\circ = \sin (45^\circ + 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\begin{aligned} \text{(iii) } \cos (90^\circ + \theta) &= \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta \\ &= (0)(\cos \theta) - (1) \sin \theta = -\sin \theta \end{aligned}$$

$$\begin{aligned} \text{(iv) } \sin (90^\circ + \theta) &= \sin 90^\circ \cos \theta + \cos 90^\circ \sin \theta \\ &= (1)(\cos \theta) + (0) \sin \theta = \cos \theta \end{aligned}$$

$$\text{(v) } \cos (105^\circ) = \cos (60^\circ + 45^\circ)$$

$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = -\left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)$$

$$= -\frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\text{(vi) } \sin (105^\circ) = \sin 60^\circ + 45^\circ$$

$$\begin{aligned}
&= \sin 60^\circ \cos 45^\circ (\cos \theta) + \cos 60^\circ \sin 45^\circ \\
&= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \\
&= -\frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

$$\begin{aligned}
3. \quad \tan(\theta_1 + \theta_2) &= \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} \\
&= \frac{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2}
\end{aligned}$$

Divide the numerator and denominator by  $\cos \theta_1 \cos \theta_2$  to obtain

$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$$

$$\begin{aligned}
4. \quad (i) \quad (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^6)(1+\omega^8) \\
&= (1+\omega)(1+\omega^2)(1+\omega)(1+1)(1+\omega^2) \\
&= 2((1+\omega)(1+\omega^2))^2 = 2((-\omega^2)(-\omega))^2 = 2\omega^6 = 2
\end{aligned}$$

$$\begin{aligned}
(ii) \quad (1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 \\
&= (-\omega-\omega)^5 + (-\omega^2-\omega^2)^5 \\
&= (-2)^5\omega^5 + (-2)^5(\omega^2)^5 \\
&= -32\omega^2 - 32\omega = -32(\omega^2 + \omega) \\
&= (-32)(-1) = 32
\end{aligned}$$

$$\begin{aligned}
(iii) \quad (2+3\omega+2\omega^2)^9 \\
&= (2+2\omega+2\omega^2+\omega)^9 = (0+\omega)^9 = \omega^9 = 1 \\
&\text{and } (2+2\omega+2\omega^2+\omega^2)^9 = (2+2\omega+2\omega^2+\omega^2)^9 \\
&= (0+\omega^2)^9 = \omega^{18} = 1
\end{aligned}$$

$$\begin{aligned}
5. \quad (i) \quad x+y+z &= a(1+\omega^2+\omega) + b(1+\omega^2+\omega) \\
&= (0) + b(0) = 0
\end{aligned}$$

$$(ii) \quad x^2+y^2+z^2$$

$$\begin{aligned}
&= (a^2 + b^2 + 2ab) + (a^2\omega^2 + b^2\omega^4 + 2ab\omega^3) + (a^2\omega^4 + b^2\omega^2 + 2ab\omega^3) \\
&= a^2(1 + \omega^2 + \omega^2) + b^2(1 + \omega^4 + \omega^2) + 2ab(1 + \omega^3 + \omega^3) \\
&= a^2(0) + b^2(0) + 2ab(1 + 1 + 1) = 6ab
\end{aligned}$$

We know that

$$\begin{aligned}
&x^3 + y^3 + z^3 - 3xyz \\
&= (x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy) \\
&= 0
\end{aligned}$$

$$\text{Thus, } x^3 + y^3 + z^3 = 3xyz$$

$$\text{Also, } xyz = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$$

$$= (a+b)[a^3\omega^3 + b^3\omega^3 + ab(\omega^2 + \omega^4)]$$

$$= (a+b)(a^2 + b^2 - ab) = a^3 + b^3$$

Thus,

$$x^3 + y^3 + z^3 - 3xyz = 3(a^3 + b^3)$$

## 2.8 SUMMARY

In this unit, first of all, in **section 2.2**, the concept of complex number is defined. In **section 2.3**, various algebraic operations, viz., addition, subtraction, multiplication and division of two complex numbers are defined and illustrated with suitable examples. In **section 2.4**, concepts of conjugate of a complex number and modulus of a complex number are defined and explained with suitable examples. The properties of conjugate and modulus operations are stated without proof. In **section 2.5**, representation of a complex number as a point in a plane, in cartesian and polar forms, are explained. Finally, in **section 2.6**, DeMoivre's Theorem for integral index, for finding nth power of a complex number, is illustrated with a number of examples. Also, some properties of cube roots of unity are discussed.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.7**.